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2001 J. Phys. A: Math. Gen. 34 7917

(http://iopscience.iop.org/0305-4470/34/39/301)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 7917–7932 PII: S0305-4470(01)26960-1

Correlations for the Cauchy and generalized circular ensembles with orthogonal and symplectic symmetry

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Received 16 July 2001
Published 21 September 2001
Online at stacks.iop.org/JPhysA/34/7917

Abstract
The generalized circular ensemble, which specifies a spectrum singularity in random matrix theory, is equivalent to the Cauchy ensemble via a stereographic projection. The Cauchy weight function is classical, and as such the n-point distribution function in the cases of orthogonal and symplectic symmetry have expressions in terms of quaternion determinants with elements given in an explicit form suitable for asymptotic analysis. The asymptotic analysis is undertaken in the neighbourhood of the spectrum singularity in both cases, and it is shown that each quaternion determinant is specified by a single function involving Bessel functions.

PACS numbers: 05.45.-a, 02.10.Yn, 02.30.Gp, 02.40.-k

1. Introduction

Random matrix ensembles specified by the eigenvalue probability density function (PDF) proportional to

$$\prod_{j=1}^{N} \mathrm{e}^{-cx_j^2} x_j^{\chi} \prod_{k=1}^{p} (x_j^2 + m_k^2) \prod_{1 \leqslant j < k \leqslant N} |x_k - x_j|^{\beta} \tag{1.1}$$

occur in statistical studies of QCD in three dimensions [16]. The m_k are masses, the parameter c determines the scale while $\chi = 0, 1$ or 2 depending on the parity of p and β. The latter parameter takes on the values $\beta = 1$ (orthogonal symmetry), $\beta = 2$ (unitary symmetry) or $\beta = 4$ (symplectic symmetry). For finite N, the general n-point distribution of (1.1) in the orthogonal and symplectic cases has recently been computed in terms of Pfaffians by Nagao and Nishigaki [13]. Also considered were various $N \to \infty$ scaled limits. A different approach to this problem has been presented by Hilmoine and Niclasen [9], in which formulae for the finite N n-point distribution function were given in the massless case $m_k = 0$ $(k = 1, \ldots, p)$.

Subsequently, Abild-Pedersen and Vernizzi [1] have extended the methods of [9] to the massive case.

In the massless case with χ even, the PDF (1.1) has the form

$$\frac{1}{C}\prod_{j=1}^{N} \mathrm{e}^{-cx_j^2}|x_j|^{\beta a} \prod_{1\leqslant j<k\leqslant N} |x_k - x_j|^{\beta}. \tag{1.2}$$

Note that with $a = 1$ this has the interpretation as a Gaussian random matrix ensemble conditioned so that there is a zero eigenvalue. Similarly, for general $a \in Z^+$, (1.2) gives the (renormalized) PDF for the Gaussian random matrix ensemble with an a fold degeneracy at the origin. For this reason we will refer to the neighbourhood of $x = 0$ in the ensemble (1.2) as the neighbourhood of the spectrum singularity. The n-point distribution function of (1.2) in the unitary symmetry case ($\beta = 2$) has been computed by Nagao and Slevin [14], as has the $N \to \infty$ scaled limit of the spectrum singularity in the neighbourhood of the spectrum singularity. The latter is given by

$$\rho_n^{\text{s.s.}}(x_1, \dots, x_n) = \det\left[K^{\text{s.s.}}(x_j, x_k)\right]_{j,k=1,\dots,n} \tag{1.3}$$

$$K^{\text{s.s.}}(x, y) := (\pi x)^{1/2}(\pi y)^{1/2}\frac{\left(J_{a+1/2}(\pi x)J_{a-1/2}(\pi y) - J_{a+1/2}(\pi y)J_{a-1/2}(\pi x)\right)}{2(x-y)} \tag{1.4}$$

which is valid for all values of $x_1, \dots, x_n$ in the case a a non-negative integer, while it is valid for $x_1, \dots, x_n$ all positive otherwise.

The objective of this paper is to compute the scaled limit in the neighbourhood of the spectrum singularity of the n-point distribution corresponding to (1.2) in the orthogonal and symplectic cases. This task was left unsolved in [9], as the finite N results therein were not amenable to asymptotic analysis. Also, the results of [13] for the scaled $N \to \infty$ limit of (1.1) with $m_k = 0$ gives formulae more complicated (involving Pfaffians whose dimension is proportional to a) and less general (requiring a to be twice a non-negative integer in the case $\beta = 1$, and a to be a non-negative integer or half integer in the case $\beta = 4$) than those to be presented here.

The difficulties faced in the calculations of [9] can be understood as resulting from the non-classical nature of the weight function

$$w(x) = \mathrm{e}^{-cx^2}|x|^{\beta a} \tag{1.5}$$

in (1.2). We recall [2] that a weight function $w(x)$ is called classical if its logarithmic derivative, written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\log w(x) = -\frac{g(x)}{f(x)} \tag{1.6}$$

is such that f and g are polynomials with

$$\text{degree } f \leqslant 2 \qquad \text{degree } g \leqslant 1. \tag{1.7}$$

In the case of the weight (1.5) the formula (1.6) holds with degree $g = 2$ for $a \neq 0$. Although the n-point distribution for general matrix ensembles with orthogonal and symplectic symmetry can be expressed as a quaternion determinant, only in the classical cases can the elements of the quaternion determinant be written in an explicit form suitable for asymptotic analysis [2].

We avoid the complications inherent with the non-classical weight function (1.2) by a combination of two ideas. The first is to notice that the scaled limit of the ensemble (1.2) in the neighbourhood of the spectrum singularity at $x = 0$ will be identical to the scaled limit of the PDF

$$\frac{1}{C}\prod_{j=1}^{N} |1 - \mathrm{e}^{\mathrm{i}\theta_j}|^{\beta a} \prod_{1\leqslant j<k\leqslant N} |\mathrm{e}^{\mathrm{i}\theta_k} - \mathrm{e}^{\mathrm{i}\theta_j}|^{\beta} \tag{1.8}$$

in the neighbourhood of the spectrum singularity at $\theta = 0$. In the special case $\beta = 2$ this has been checked by explicit calculation [12], while its validity for general β is expected because both PDFs are identical in the neighbourhood of the respective spectrum singularities. The second idea is to transform (1.8) from an eigenvalue PDF on the circle to an eigenvalue PDF on the line via the stereographic projection

$$e^{i\theta_j} = \frac{1 - ix_j}{1 + ix_j}. \tag{1.9}$$

This shows

$$\prod_{j=1}^{N} |1 - e^{i\theta_j}|^{\beta a} \prod_{1 \leqslant j<k \leqslant N} |e^{i\theta_k} - e^{i\theta_j}|^{\beta} \, d\theta_1 \cdots d\theta_N$$
$$\propto \prod_{j=1}^{N} (1 + x_j^2)^{-\beta(N+a-1)/2-1} \prod_{1 \leqslant j<k \leqslant N} |x_k - x_j|^{\beta} \, dx_1 \cdots dx_N. \tag{1.10}$$

Now the weight function occurring on the right-hand side of (1.10) has the form

$$w(x) = (1 + x^2)^{-\mu} \tag{1.11}$$

which defines the Cauchy weight. Written in the form (1.6) it gives

$$g(x) = 2\mu x \qquad f(x) = 1 + x^2.$$

These polynomials have the property (1.7) so the Cauchy weight is classical. Consequently the general formulae of [2] are applicable, and it is these formulae which allow the scaled $N \to \infty$ limit to be computed.

We will proceed by revising the general formulae of [2] for the n-point distribution $\rho_n^{\mathrm{Cy}}(x_1, \ldots, x_n)$ in the orthogonal and symplectic ensembles with Cauchy weight. This is done in section 2. In section 3 we use the fact that according to the transformation (1.9), and with $\theta_j \mapsto 2\pi X_j/N$ in (1.8) (the mean density in the variable X_j is unity by this scaling), the n-point distribution $\rho_n^{\mathrm{GC}}(X_1, \ldots, X_n)$ in the generalized circular ensemble is related to that in the Cauchy ensemble by

$$\rho_n^{\mathrm{GC}}(X_1, \ldots, X_n) = \prod_{j=1}^{n} \frac{4\pi}{|1 + z_j|^2 N} \rho_n^{\mathrm{Cy}}(x_1, \ldots, x_n) \Bigg|_{\substack{x_j = i(1-z_j)/(1+z_j) \\ \mu = \beta(N+a-1)/2+1}} \tag{1.12}$$

where

$$z_j := e^{2\pi i X_j/N} \tag{1.13}$$

to then compute the scaled $N \to \infty$ limit in the neighbourhood of the spectrum singularity. The latter occurs at $z_j = -1$ so we make the replacements $X_j \mapsto X_j + N/2$ or equivalently $z_j \mapsto -z_j$ and rewrite (1.12) as

$$\rho_n^{\mathrm{GC}}(X_1, \ldots, X_n) = \prod_{j=1}^{n} \frac{4\pi}{|1 - z_j|^2 N} \rho_n^{\mathrm{Cy}}(x_1, \ldots, x_n) \Bigg|_{\substack{x_j = i(1+z_j)/(1-z_j) \\ \mu = \beta(N+a-1)/2+1}} \tag{1.14}$$

with our objective being to compute the $N \to \infty$ limit, with X_j fixed, of the right-hand side of this expression. In both the orthogonal and symplectic symmetry cases ρ_n^{CG} is evaluated as a $n \times n$ quaternion determinant. This is given by (3.34) in the case of orthogonal symmetry, with $S_1^{\mathrm{s.s.}}$ therein specified by (3.26) or equivalently (3.30), while in the case of symplectic symmetry it is given by (3.47) with $S_4^{\mathrm{s.s.}}$ therein specified by (3.44) (the quantity $K^{\mathrm{s.s.}}$ in (3.26), (3.30) and (3.44) is specified by (1.4)). Properties of ρ_n^{GC} are discussed in section 4. These include the relationship with the distributions for the Dyson circular ensembles, the connection with the soft edge distributions in a scaled $a \to \infty$ limit, and a sum rule obeyed by the density.

2. n-point distribution for classical ensembles

2.1. Orthogonal symmetry

The eigenvalue PDF for a matrix ensemble with orthogonal symmetry has the general form

$$p_1(x_1, \ldots, x_N) := \frac{1}{\hat{Z}_N} \prod_{j=1}^{N} w_1(x_j) \prod_{1 \leqslant j < k \leqslant N} |x_k - x_j|$$
$$\hat{Z}_N := \int_{-\infty}^{\infty} dx_1\, w_1(x_1) \cdots \int_{-\infty}^{\infty} dx_N\, w_1(x_N) \prod_{1 \leqslant j < k \leqslant N} |x_k - x_j|.$$

The weights

$$w_1(x) = \begin{cases} e^{-x^2/2} & \text{Gaussian} \\ x^{(a-1)/2} e^{-x/2}\ (x > 0) & \text{Laguerre} \\ (1-x)^{(a-1)/2}(1+x)^{(b-1)/2}\ (-1 < x < 1) & \text{Jacobi} \\ (1+x^2)^{-(\alpha+1)/2} & \text{Cauchy} \end{cases} \tag{2.1}$$

specify the classical weights. In the classical cases the n-point distribution function

$$\rho_n(x_1, \ldots, x_n) = N(N-1)\cdots(N-n+1) \int_{-\infty}^{\infty} dx_{n+1} \cdots \int_{-\infty}^{\infty} dx_N\, p_1(x_1, \ldots, x_N) \tag{2.2}$$

can be expressed in terms of the monic orthogonal polynomials $\{p_k(x)\}_{k=0,1,\ldots}$ associated with the weight functions

$$w_2(x) = \begin{cases} e^{-x^2} & \text{Gaussian} \\ x^a e^{-x}\ (x > 0) & \text{Laguerre} \\ (1-x)^a(1+x)^b\ (-1 < x < 1) & \text{Jacobi} \\ (1+x^2)^{-\alpha} & \text{Cauchy} \end{cases} \tag{2.3}$$

and their corresponding normalizations. For future reference we note that in the Cauchy case the monic orthogonal polynomials, $p_k^{\rm Cy}(x)$ say, are given in terms of the monic Jacobi polynomials, $p_k^{\rm J}(x)$ say, by (see e.g. [18])

$$p_k^{\rm Cy}(x) := i^{-k} p_k^{\rm J}(x)\big|_{a=b=-\alpha} \qquad k < \alpha \tag{2.4}$$

(the bound on k is required because the Cauchy weight only has a finite number of well defined moments). Also, the normalization associated with the polynomials (2.4), $(p_k, p_k)_2^{\rm Cy}$ say, has the explicit form (see e.g. [18])

$$(p_k, p_k)_2^{\rm Cy} = \pi 2^{-2(\alpha-k-1)} \frac{\Gamma(k+1)\Gamma(2\alpha-2k)\Gamma(2\alpha-1-2k)}{\Gamma(2\alpha-k)(\Gamma(\alpha-k))^2}. \tag{2.5}$$

In the classical cases the n-point distribution function is expressed in terms of a quaternion determinant formula [2]

$$\rho_n(x_1, \ldots, x_n) = \text{qdet}[f_1(x_j, x_k)]_{j,k=1,\ldots,n}$$
$$f_1(x, y) = \begin{bmatrix} S_1(x, y) & I_1(x, y) \\ D_1(x, y) & S_1(y, x) \end{bmatrix} \tag{2.6}$$

where with

$$P_n(x, y) := \big(w_2(x) w_2(y)\big)^{1/2} \sum_{k=0}^{n-1} \frac{p_k(x) p_k(y)}{(p_k, p_k)_2} \tag{2.7}$$

we have

$$S_1(x,y) = \frac{(w_2(x))^{1/2}}{w_1(x)} \frac{w_1(y)}{(w_2(y))^{1/2}} P_{N-1}(x,y) + \gamma_{N-2} w_1(y) p_{N-1}(y) \times \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{sgn}(x-t) p_{N-2}(t) w_1(t)\, \mathrm{d}t \tag{2.8}$$

and

$$I_1(x,y) = -\int_x^y S_1(x,z)\,\mathrm{d}z - \tfrac{1}{2}\mathrm{sgn}(x-y) \qquad D_1(x,y) = \frac{\partial}{\partial x} S_1(x,y). \tag{2.9}$$

In (2.8) it is assumed N is even (an analogous formula is known for N odd, but since our interest is in the limit $N \to \infty$ the N even case suffices) and

$$\gamma_k := \frac{1}{(p_k, p_k)_2} \begin{cases} 1 & \text{Hermite} \\ \frac{1}{2} & \text{Laguerre} \\ \frac{1}{2}(2k+2+a+b) & \text{Jacobi} \\ \alpha - 1 - k & \text{Cauchy.} \end{cases} \tag{2.10}$$

2.2. *Symplectic symmetry*

In the symplectic case, the eigenvalue PDF has the general form

$$p_4(x_1,\ldots,x_N) := \frac{1}{\hat{Z}_N} \prod_{j=1}^N w_4(x_j) \prod_{1\leqslant j<k\leqslant N} |x_k - x_j|^4$$

$$\hat{Z}_N := \int_{-\infty}^{\infty} \mathrm{d}x_1\, w_4(x_1) \cdots \int_{-\infty}^{\infty} \mathrm{d}x_N\, w_4(x_N) \prod_{1\leqslant j<k\leqslant N} |x_k - x_j|^4$$

and the classical weights are specified by

$$w_4(x) = \begin{cases} \mathrm{e}^{-x^2} & \text{Gaussian} \\ x^{a+1}\mathrm{e}^{-x}\ (x>0) & \text{Laguerre} \\ (1-x)^{a+1}(1+x)^{b+1}\ (-1<x<1) & \text{Jacobi} \\ (1+x^2)^{-\alpha+1} & \text{Cauchy.} \end{cases} \tag{2.11}$$

The n-point distribution function is specified by (2.2) with p_4 replacing p_1.

In the classical cases the n-point distribution function has an evaluation in terms of a quaternion determinant analogous to (2.6). Explicitly [2]

$$\rho_n(x_1,\ldots,x_n) = \mathrm{qdet}[f_4(x_j,x_k)]_{j,k=1,\ldots,n} \qquad f_4(x,y) = \begin{bmatrix} S_4(x,y) & I_4(x,y) \\ D_4(x,y) & S_4(y,x) \end{bmatrix} \tag{2.12}$$

where

$$S_4(x,y) = \frac{1}{2}\left(\frac{w_2(y)}{w_4(y)}\right)^{1/2} \left(\frac{w_4(x)}{w_2(x)}\right)^{1/2} P_{2N}(x,y) - \frac{1}{2}\gamma_{2N-1} \frac{w_2(y)}{(w_4(y))^{1/2}} p_{2N}(y) \int_x^{\infty} p_{2N-1}(t) \frac{w_2(t)}{(w_4(t))^{1/2}}\,\mathrm{d}t \tag{2.13}$$

and

$$I_4(x,y) = -\int_x^y S_4(x,y')\,\mathrm{d}y' \qquad D_4(x,y) = \frac{\partial}{\partial x} S_4(x,y). \tag{2.14}$$

3. Scaled $N \to \infty$ limit about the spectrum singularity

The task now is to compute the $N \to \infty$ limit of (1.14) with ρ_k given by (2.6) (orthogonal case) and (2.12) (symplectic case), and the quantities therein specified as required for the Cauchy weight. In both cases we are to substitute $x_j = \mathrm{i}(1+z_j)/(1-z_j) = -\cot \pi X_j/N$. To analyse this limit the first task is to express the polynomial $p_k^{\mathrm{Cy}}(x)$, specified in terms of a monic Jacobi polynomial by (2.4), in a suitable form. For this purpose we use the formula [4]

$$P_n^{(\alpha,\beta)}(x) \propto \left(\frac{x-1}{2}\right)^n {}_2F_1\left(-n, -n-\alpha; -2n-\alpha-\beta; -\frac{2}{x-1}\right)$$

where ${}_2F_1$ denotes the Gauss hypergeometric function. Substituting in (2.4) gives

$$p_k^{\mathrm{Cy}}\left(\mathrm{i}\frac{1+z}{1-z}\right) = i^{-k}\left(-\frac{2}{1-z}\right)^k {}_2F_1(-k, -k+\alpha; -2k+2\alpha; 1-z) \tag{3.1}$$

where use has been make of the fact that by the requirement that $p_k^{\mathrm{Cy}}(z)$ be monic, the coefficient of $2\mathrm{i}/(1-z)$ on the right-hand side must be unity. We note too that the summation (2.7) defining P_n can be summed for general monic orthogonal polynomials according to the Christoffel–Darboux formula. This gives [15]

$$P_n(x,y) = (w_2(x)w_2(y))^{1/2}\frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{(p_{n-1}, p_{n-1})_2(x-y)}. \tag{3.2}$$

To proceed further the orthogonal and symplectic cases must be treated separately.

3.1. Orthogonal symmetry

In (2.1) the Cauchy weight function is specified as $w_1(x) = (1+x^2)^{-(\alpha+1)/2}$. Comparing with (1.10) in the case $\beta = 1$ shows we require

$$\alpha = N + a. \tag{3.3}$$

With α so specified, consider now the computation of the scaled form of

$$P_{N-1}\left(\mathrm{i}\frac{1+z}{1-z}, \mathrm{i}\frac{1+w}{1-w}\right)\bigg|_{\alpha=N+a} \qquad z := \mathrm{e}^{2\pi \mathrm{i}X/N} \quad w := \mathrm{e}^{2\pi \mathrm{i}Y/N}$$

which from (3.2), (3.1) and (3.3) requires the scaled form of

$$p_{N-1}^{\mathrm{Cy}}\left(\mathrm{i}\frac{1+z}{1-z}\right)\bigg|_{\alpha=N+a} = i^{-(N-1)}\left(-\frac{2}{1-z}\right)^{N-1} {}_2F_1(-(N-1), a+1; 2a+2; 1-z)$$
$$p_{N-2}^{\mathrm{Cy}}\left(\mathrm{i}\frac{1+z}{1-z}\right)\bigg|_{\alpha=N+a} = i^{-(N-2)}\left(-\frac{2}{1-z}\right)^{N-2} {}_2F_1(-(N-2), a+2; 2a+4; 1-z).$$

The scaled form of the latter follow from the formulae [4]

$$\lim_{n\to\infty} {}_2F_1(-n, b; c; t/n) = {}_1F_1(b; c; -t) \tag{3.4}$$

$${}_1F_1(a; 2a; 2\mathrm{i}x) = \Gamma(a+1/2)(x/2)^{-(a-1/2)}\mathrm{e}^{\mathrm{i}x}J_{a-1/2}(x) \tag{3.5}$$

which give

$$p_{N-1}^{\mathrm{Cy}}\left(\mathrm{i}\frac{1+z}{1-z}\right)\bigg|_{\alpha=N+a} \sim \frac{(-1)^{N-1}}{(\sin \pi X/N)^{N-1}}\Gamma(a+3/2)(\pi X/2)^{-(a+1/2)}J_{a+1/2}(\pi X) \tag{3.6}$$

$$p_{N-1}^{\mathrm{Cy}}\left(\mathrm{i}\frac{1+z}{1-z}\right)\bigg|_{\alpha=N+a} \sim \frac{(-1)^{N-2}}{(\sin \pi X/N)^{N-2}}\Gamma(a+5/2)(\pi X/2)^{-(a+3/2)}J_{a+3/2}(\pi X). \tag{3.7}$$

It follows from these results and the simple formula

$$\frac{1}{x-y} \sim \frac{\pi}{N} \frac{XY}{X-Y} \tag{3.8}$$

that

$$\begin{aligned} &\frac{p_{N-1}^{\mathrm{Cy}}(x)p_{N-2}^{\mathrm{Cy}}(y) - p_{N-1}^{\mathrm{Cy}}(y)p_{N-2}^{\mathrm{Cy}}(x)}{x-y} \\ &\quad \sim -\frac{\Gamma(a+3/2)\Gamma(a+5/2)}{(\pi X/N)^{N-1}(\pi Y/N)^{N-1}}(\pi X/2)^{-(a+1/2)}(\pi Y/2)^{-(a+1/2)} \\ &\quad \times \left(\frac{2}{N}\right) XY \frac{\{J_{a+1/2}(\pi X)J_{a+3/2}(\pi Y) - J_{a+1/2}(\pi Y)J_{a+3/2}(\pi X)\}}{X-Y}. \end{aligned}$$

Also required is the normalization in (3.2) in the case $n = N-1$, which according to (2.5) is given by

$$(p_{N-2}, p_{N-2})_2^{\mathrm{Cy}}\Big|_{\alpha=N+a} = \pi 2^{-2(a+1)} \frac{\Gamma(N-1)\Gamma(2a+4)\Gamma(2a+3)}{\Gamma(N+2a+2)(\Gamma(a+2))^2}.$$

Noting from the duplication formula for the gamma function that

$$\Gamma(a+3/2)\Gamma(a+5/2) = 2^{-4a-5}\pi \frac{\Gamma(2a+3)\Gamma(2a+4)}{(\Gamma(a+2))^2} \tag{3.9}$$

we therefore have

$$\frac{\Gamma(a+3/2)\Gamma(a+5/2)}{(p_{N-2}, p_{N-2})_2^{\mathrm{Cy}}\Big|_{\alpha=N+a}} = 2^{-2a-3} \frac{\Gamma(N+2a+2)}{\Gamma(N-1)} \sim 2^{-2a-3} N^{2a+3}. \tag{3.10}$$

Finally we note that

$$\begin{aligned} &\left(w_2\left(\mathrm{i}\frac{1+z}{1-z}\right) w_2\left(\mathrm{i}\frac{1+w}{1-w}\right)\right)^{1/2}\Bigg|_{\alpha=N+a} = \left(\sin\frac{\pi X}{N}\right)^{N+a} \left(\sin\frac{\pi Y}{N}\right)^{N+a} \\ &\frac{(w_2(x))^{1/2} w_1(y)}{w_1(x)(w_2(x))^{1/2}} \sim \left|\frac{Y}{X}\right|. \end{aligned} \tag{3.11}$$

Combining the above results shows

$$\frac{(w_2(x))^{1/2} w_1(y)}{w_1(x)(w_2(x))^{1/2}} P_{N-1}(x, y) \sim \frac{\pi}{N} Y^2 K^{\mathrm{s.s.}}(X, Y)\Big|_{a\mapsto a+1} \tag{3.12}$$

where $K^{\mathrm{s.s.}}$ is specified by (1.4).

We turn our attention to the calculation of the scaled form of the second term on the right-hand side of (2.8), considering each factor in turn. Now, according to (2.10), (3.3) and (3.10),

$$\gamma_{N-2} = \frac{a+1}{(p_{N-2}, p_{N-2})_2^{\mathrm{Cy}}} \sim (a+1) \frac{2^{-2a-3} N^{2a+3}}{\Gamma(a+3/2)\Gamma(a+5/2)}. \tag{3.13}$$

Recalling (2.1) and (3.3), and setting

$$y = \mathrm{i}\frac{1-w}{1+w} \qquad w := \mathrm{e}^{2\pi \mathrm{i} Y/N}$$

we see that

$$w_1(y) = (1+y^2)^{-(N+a+1)/2} \sim \left(\frac{\pi Y}{N}\right)^{N+a+1}.$$

The large-N behaviour of $p_{N-1}(y)$ is given by (3.6) with X replaced by Y. Regarding the integral we write

$$\int_{-\infty}^{\infty} \mathrm{sgn}(x-t) p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t = 2\int_{-x}^{\infty} p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t - \int_{-\infty}^{\infty} p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t. \tag{3.14}$$

But we know from [5] that for the classical weights (2.3), with N even,

$$\int_{-\infty}^{\infty} p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t = \left(\int_{-\infty}^{\infty} w_1(t)\,\mathrm{d}t\right)\prod_{j=0}^{N/2-2}\frac{\gamma_{2j}}{\gamma_{2j+1}} \tag{3.15}$$

where γ_j are given by (2.10). For the first factor on the right-hand side of (3.15) we use the definite integral

$$\int_{-\infty}^{\infty}\frac{x^{2q}}{(1+x^2)^{\alpha}}\,\mathrm{d}x = \frac{\Gamma(1/2+q)\Gamma(\alpha-q-1/2)}{\Gamma(\alpha)} \tag{3.16}$$

(which is equivalent to the Euler beta integral [17]) to deduce that

$$\int_{-\infty}^{\infty} w_1(t)\,\mathrm{d}t = \frac{\Gamma(1/2)\Gamma((N+a)/2)}{\Gamma((N+a+1)/2)} \sim \pi^{1/2}(N/2)^{-1/2}. \tag{3.17}$$

For the second factor, we note from (2.10), (3.3) and (2.5) that

$$\frac{\gamma_{2j}}{\gamma_{2j+1}} = \frac{N+a-1-2j}{N+a-2-2j}\frac{(2j+1)(2(N+a)-2j-1)}{(2(N+a)-4j-1)(2(N+a)-4j-3)} \tag{3.18}$$

and from this we deduce

$$\prod_{j=0}^{N/2-1}\frac{\gamma_{2j}}{\gamma_{2j+1}} \sim (N/2)^{-1/2}\frac{1}{(2N)^{a+1}}\frac{\Gamma(a/2+2)\Gamma(2a+4)}{\Gamma(a/2+3/2)\Gamma(a+3)}. \tag{3.19}$$

Consequently

$$\int_{-\infty}^{\infty} p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t \sim \pi^{1/2}\frac{2^{-a}}{N^{a+2}}\frac{\Gamma(a/2+2)\Gamma(2a+4)}{\Gamma(a/2+3/2)\Gamma(a+3)}. \tag{3.20}$$

It remains to compute the scaled form of the first integral in (3.14). Now

$$\begin{aligned}\int_{-x}^{\infty} p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t &= \int_{\cot\pi X/N}^{\infty} p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t \\ &= \frac{\pi}{N}\int_0^X (\sin\pi s/N)^{\alpha-1} p_{N-2}^{\mathrm{Cy}}(\cot\pi s/N)\,\mathrm{d}s \end{aligned}\tag{3.21}$$

where the second equality follows from the substitution $t = \cot\pi s/N$. Substituting for p_{N-2}^{Cy} in (3.21) according to (3.7) and recalling (3.3) shows

$$\int_{-x}^{\infty} p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t \sim \frac{\pi}{N^{a+2}}(-1)^{N-2}\Gamma(a+5/2)2^{a+3/2}\int_0^X (\pi s)^{-1/2} J_{a+3/2}(\pi s)\,\mathrm{d}s. \tag{3.22}$$

Making use of the identity

$$\Gamma(a+5/2) = 2^{-2(a+2)}\pi^{1/2}\frac{\Gamma(2a+5)}{\Gamma(a+3)}$$

which like (3.9) is a consequence of the duplication formula for the gamma function, we see from (3.20) and (3.22) substituted in (3.14) that

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{sgn}(x-t) p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t &\sim -\frac{2^{a+2}}{N^{a+2}}\frac{\Gamma(a/2+1)\Gamma(a+5/2)}{\Gamma(a/2+3/2)} \\ &\quad\times\left(1 - 2^{1/2}\frac{\Gamma(a/2+3/2)}{\Gamma(a/2+1)}\int_0^{\pi X} s^{-1/2} J_{a+3/2}(s)\,\mathrm{d}s\right).\end{aligned}\tag{3.23}$$

Combining the above results gives that

$$\gamma_{N-2} w_1(y) p_{N-1}^{\mathrm{Cy}}(y) \tfrac{1}{2} \int_{-\infty}^{\infty} \operatorname{sgn}(x-t) p_{N-2}^{\mathrm{Cy}}(t) w_1(t)\,\mathrm{d}t$$
$$\sim \left(\frac{\pi Y^2}{N}\right) \pi \frac{\Gamma(a/2+1)}{\Gamma(a/2+1/2)} \frac{J_{a+1/2}(\pi Y)}{(2\pi Y)^{1/2}}$$
$$\times \left(1 - 2^{1/2} \frac{\Gamma(a/2+3/2)}{\Gamma(a/2+1)} \int_0^{\pi X} s^{-1/2} J_{a+3/2}(s)\,\mathrm{d}s\right). \tag{3.24}$$

Adding (3.12) to this shows

$$S_1\left(\mathrm{i}\frac{1+z}{1-z}, \mathrm{i}\frac{1+w}{1-w}\right)\bigg|_{\alpha=N+a} \sim \frac{\pi Y^2}{N} S_1^{\mathrm{s.s.}}(X,Y) \tag{3.25}$$

where

$$S_1^{\mathrm{s.s.}}(X,Y) = K^{\mathrm{s.s.}}(X,Y)\Big|_{a\mapsto a+1} + \pi \frac{\Gamma(a/2+1)}{\Gamma(a/2+1/2)} \frac{J_{a+1/2}(\pi Y)}{(2\pi Y)^{1/2}}$$
$$\times \left(1 - 2^{1/2} \frac{\Gamma(a/2+3/2)}{\Gamma(a/2+1)} \int_0^{\pi X} s^{-1/2} J_{a+3/2}(s)\,\mathrm{d}s\right). \tag{3.26}$$

We note that for $a > 0$, $S_1^{\mathrm{s.s.}}$ can be written in a form analogous to (3.26), only involving $K^{\mathrm{s.s.}}$ without the replacement $a \mapsto a+1$. The first step is to use the recurrence

$$zJ_{\alpha-1}(z) - 2\alpha J_\alpha(z) + zJ_{\alpha+1}(z) = 0$$

in (1.4) to deduce

$$K^{\mathrm{s.s.}}(X,Y)\Big|_{a\mapsto a+1} = K^{\mathrm{s.s.}}(X,Y) - (a+1/2) \frac{J_{a+1/2}(\pi x) J_{a+1/2}(\pi y)}{(\pi x)^{1/2} (\pi y)^{1/2}}. \tag{3.27}$$

The second step is to use the identity

$$-z^{-\nu} J_{\nu+1}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\{z^{-\nu} J_\nu(z)\} \tag{3.28}$$

to rewrite the integral in (3.26) as

$$-\int_0^{\pi X} s^a \frac{\mathrm{d}}{\mathrm{d}s}\{s^{-a-1/2} J_{a+1/2}(s)\}\,\mathrm{d}s = -(\pi X)^{-1/2} J_{a+1/2}(\pi X) + a \int_0^{\pi X} s^{-3/2} J_{a+1/2}(s)\,\mathrm{d}s$$
$$a > 0. \tag{3.29}$$

Substituting (3.27) and (3.29) in (3.26) shows that for $a > 0$ we can write

$$S_1^{\mathrm{s.s.}}(X,Y) = K^{\mathrm{s.s.}}(X,Y) + \pi \frac{\Gamma(a/2+1)}{\Gamma(a/2+1/2)} \frac{J_{a+1/2}(\pi Y)}{(2\pi Y)^{1/2}}$$
$$\times \left(1 - 2^{3/2} \frac{\Gamma(a/2+3/2)}{\Gamma(a/2)} \int_0^{\pi X} s^{-3/2} J_{a+1/2}(s)\,\mathrm{d}s\right). \tag{3.30}$$

With the result (3.25) established, we see from (2.9) that

$$I_1\left(\mathrm{i}\frac{1+z}{1-z}, \mathrm{i}\frac{1+w}{1-w}\right)\bigg|_{\alpha=N+a} = -\int_{-\cot\pi X/N}^{-\cot\pi Y/N} S_1(-\cot\pi X/N, z)\,\mathrm{d}z - \frac{1}{2}\operatorname{sgn}(X-Y)$$
$$= -\frac{\pi}{N} \int_X^Y \frac{1}{\sin^2 \pi s/N} S_1(-\cot\pi X/N, -\cot\pi s/N)\,\mathrm{d}s - \frac{1}{2}\operatorname{sgn}(X-Y)$$
$$\sim -\int_X^Y S_1(X,s)\,\mathrm{d}s - \tfrac{1}{2}\operatorname{sgn}(X-Y) \tag{3.31}$$

where the second equality follows from the change of variable $z = -\cot \pi s/N$, and

$$D_1\left(\mathrm{i}\frac{1+z}{1-z}, \mathrm{i}\frac{1+w}{1-w}\right)\bigg|_{\alpha=N+a} = -\frac{\partial}{\partial \cot \pi X/N} S_1\left(\mathrm{i}\frac{1+z}{1-z}, \mathrm{i}\frac{1+w}{1-w}\right) \sim \left(\frac{\pi XY}{N}\right)^2 \frac{\partial}{\partial X} S_1^{\mathrm{s.s.}}(X, Y). \tag{3.32}$$

The results (3.26)–(3.32) imply that the scaled form of the n-point distribution (2.6) in the Cauchy ensemble with orthogonal symmetry is given by

$$\rho_n^{\mathrm{Cy}}(X_1, \ldots, X_n) \sim \left(\frac{\pi}{N}\right)^n \prod_{j=1}^n X_j^2 \times \mathrm{qdet}\begin{bmatrix} S_1^{\mathrm{s.s.}}(X_j, X_k) & -\int_{X_j}^{X_k} S_1^{\mathrm{s.s.}}(X_j, u)\,\mathrm{d}u - \frac{1}{2}\mathrm{sgn}(X_j - X_k) \\ \frac{\partial}{\partial X_j} S_1^{\mathrm{s.s.}}(X_j, X_k) & S_1^{\mathrm{s.s.}}(X_k, X_j) \end{bmatrix}_{j,k=1,\ldots,n} \tag{3.33}$$

where a factor $\pi X_k^2/N$ has been removed from each odd numbered column and a factor of $\pi X_j^2/N$ removed from each even numbered row. Substituting this result in (1.14) gives that for the circular ensemble with orthogonal symmetry the n-point distribution function in the neighbourhood of the spectrum singularity is given by

$$\rho_n^{\mathrm{GC}}(X_1, \ldots, X_n) = \mathrm{qdet}\begin{bmatrix} S_1^{\mathrm{s.s.}}(X_j, X_k) & -\int_{X_j}^{X_k} S_1^{\mathrm{s.s.}}(X_j, u)\,\mathrm{d}u - \frac{1}{2}\mathrm{sgn}(X_j - X_k) \\ \frac{\partial}{\partial X_j} S_1^{\mathrm{s.s.}}(X_j, X_k) & S_1^{\mathrm{s.s.}}(X_k, X_j) \end{bmatrix}_{j,k=1,\ldots,n}. \tag{3.34}$$

3.2. Symplectic symmetry

Comparing the definition of the Cauchy weight in (2.11) with the weight in (1.10) evaluated at $\beta = 4$ shows we require

$$\alpha = 2(N+a). \tag{3.35}$$

We now proceed in an analogous fashion to the analysis of the $N \to \infty$ scaled limit in the orthogonal case and first consider the scaled limit of the term involving

$$P_{2N}\left(\mathrm{i}\frac{1+z}{1-z}, \mathrm{i}\frac{1+w}{1-w}\right)\bigg|_{\alpha=2(N+a)} \tag{3.36}$$

in (2.13). The asymptotic form of the polynomials p_{2N}^{Cy} and p_{2N-1}^{Cy} occurring in the summation formula (3.2) for (3.36) is deduced from (3.1) and (3.4) to be given by

$$p_{2N}^{\mathrm{Cy}}\left(\mathrm{i}\frac{1-z}{1+z}\right)\bigg|_{\alpha=2(N+a)} \sim \frac{1}{(\sin \pi X/N)^{2N}} \Gamma(2a+1/2)(\pi X)^{-(2a-1/2)} J_{2a-1/2}(2\pi X) \tag{3.37}$$

$$p_{2N-1}^{\mathrm{Cy}}\left(\mathrm{i}\frac{1-z}{1+z}\right)\bigg|_{\alpha=2(N+a)} \sim -\frac{1}{(\sin \pi X/N)^{2N-1}} \Gamma(2a+3/2)(\pi X)^{-(2a+1/2)} J_{2a+1/2}(2\pi X) \tag{3.38}$$

while from (2.5) the normalization in the summation formula is such that

$$\frac{1}{(p_{2N-1}, p_{2N-1})_2^{\mathrm{Cy}}}\bigg|_{\alpha=2(N+a)} \sim \frac{1}{\pi} 2^{2a} (2N)^{4a+1} \frac{(\Gamma(2a+1))^2}{\Gamma(4a+1)\Gamma(4a+2)} = N^{4a+1} \frac{1}{\Gamma(2a+3/2)\Gamma(2a+1/2)} \tag{3.39}$$

where the final equality follows upon use of the duplication formula for the gamma function. We also have

$$(w_2(x)w_2(y))^{1/2}\Big|_{\alpha=2(N+a)} = (\sin\pi X/N)^{2(N+a)}(\sin\pi Y/N)^{2(N+a)}$$

while after noting that for the classical weights

$$\left(\frac{w_2(y)}{w_4(y)}\right)^{1/2}\left(\frac{w_4(x)}{w_2(x)}\right)^{1/2} = \frac{(w_2(x))^{1/2}}{w_1(x)}\frac{w_1(y)}{(w_2(y))^{1/2}}$$

we see that the asymptotic form is given by the second formula in (3.11), and we can again use (3.8). Combining these results shows

$$\frac{1}{2}\left(\frac{w_2(y)}{w_4(y)}\right)^{1/2}\left(\frac{w_4(x)}{w_2(x)}\right)^{1/2} P_{2N}(x,y) \sim \frac{\pi Y^2}{N}K^{\text{s.s.}}(2X,2Y)\Big|_{a\mapsto 2a}. \tag{3.40}$$

The second term in (2.13), which has an analogous structure to the second term in (2.8), consists of a number of factors. We proceed to compute the scaled $N\to\infty$ behaviour of each factor in turn, as we did for the second term in (2.8). First we note from (2.10), (3.35) and (3.39) that

$$\gamma_{2N-1} \sim \frac{aN^{4a+1}}{\Gamma(2a+3/2)\Gamma(2a+1/2)}.$$

Next, according to (2.3) and (2.11)

$$\frac{w_2(y)}{(w_4(y))^{1/2}} = (1+y^2)^{-(N+a+1/2)} \sim \left(\frac{\pi Y}{N}\right)^{2(N+a)+1} \tag{3.41}$$

while the large-N behaviour of $p_{2N}(y)$ is given by (3.37) with $X\mapsto Y$. To analyse the integral in (2.13) we first note

$$\begin{aligned}\int_x^\infty \frac{w_2(t)}{(w_4(t))^{1/2}}p_{2N-1}^{\text{Cy}}(t)\,\mathrm{d}t &= \left(\int_{-\infty}^\infty - \int_{-\infty}^x\right)\frac{w_2(t)}{(w_4(t))^{1/2}}p_{2N-1}^{\text{Cy}}(t)\,\mathrm{d}t \\ &= -\int_{-\infty}^x \frac{w_2(t)}{(w_4(t))^{1/2}}p_{2N-1}^{\text{Cy}}(t)\,\mathrm{d}t \\ &= \int_{\cot\pi X/N}^\infty p_{2N-1}^{\text{Cy}}(t)\frac{\mathrm{d}t}{(1+t^2)^{(N+a+1/2)}}\end{aligned} \tag{3.42}$$

where in going from the first equality to the second we have used the fact that the definite integral vanishes since the integrand is odd. Changing variables $t=\cot\pi s/N$, substituting for p_{2N-1}^{Cy} according to (3.38) and making use of (3.41) shows

$$\int_x^\infty \frac{w_2(t)}{(w_4(t))^{1/2}}p_{2N-1}^{\text{Cy}}(t)\,\mathrm{d}t \sim \frac{\Gamma(2a+3/2)}{\pi^{1/2}N^{2a+1}}\int_0^{\pi X} s^{-1/2}J_{2a+1/2}(2s)\,\mathrm{d}s.$$

Multiplying the above results to form the second term in (2.8), and simplifying using the identity

$$\Gamma(2a+3/2)\Gamma(2a+1/2) = \frac{2^{-8a-1}\pi\Gamma(4a+1)\Gamma(4a+2)}{(\Gamma(2a+1))^2}$$

shows

$$\begin{aligned}&\frac{1}{2}\gamma_{2N-1}\frac{w_2(y)}{(w_4(y))^{1/2}}p_{2N}^{\text{Cy}}(y)\int_x^\infty \frac{w_2(t)}{(w_4(t))^{1/2}}p_{2N-1}^{\text{Cy}}(t)\,\mathrm{d}t \\ &\quad\sim \frac{a}{N}(\pi Y)^{3/2}J_{2a-1/2}(2\pi Y)\int_0^{\pi X} s^{-1/2}J_{2a+1/2}(2s)\,\mathrm{d}s.\end{aligned}$$

Adding this result to (3.40) as required by (2.8) we conclude

$$S_4\left(\mathrm{i}\frac{1+z}{1-z},\mathrm{i}\frac{1+w}{1-w}\right)\Big|_{\alpha=2(N+a)} \sim \frac{\pi Y^2}{N} S_4^{\mathrm{s.s.}}(X,Y) \tag{3.43}$$

where

$$S_4^{\mathrm{s.s.}}(X,Y) := K^{\mathrm{s.s.}}(2X,2Y)\big|_{a\mapsto 2a} - a\pi\frac{J_{2a-1/2}(2\pi Y)}{(\pi Y)^{1/2}}\int_0^{\pi X} s^{-1/2}J_{2a+1/2}(2s)\,\mathrm{d}s. \tag{3.44}$$

Recalling the definitions (2.14), we see from the asymptotic formula (3.43) that

$$\begin{aligned} I_4\left(\mathrm{i}\frac{1+z}{1-z},\mathrm{i}\frac{1+w}{1-w}\right)\Big|_{\alpha=2(N+a)} &\sim -\int_X^Y S_4^{\mathrm{s.s.}}(X,u)\,\mathrm{d}u \\ D_4\left(\mathrm{i}\frac{1+z}{1-z},\mathrm{i}\frac{1+w}{1-w}\right) &\sim \left(\frac{\pi XY}{N}\right)^2\frac{\partial}{\partial X}S_4^{\mathrm{s.s.}}(X,Y) \end{aligned} \tag{3.45}$$

(cf (3.31) and (3.32)).

The results (3.43) and (3.45) substituted in (2.6) show that the scaled form of the n-point distribution for the Cauchy ensemble with symplectic symmetry is given by

$$\rho_n^{\mathrm{Cy}}(X_1,\ldots,X_n) \sim \left(\frac{\pi}{N}\right)^n\prod_{j=1}^n X_j^2\,\mathrm{qdet}\begin{bmatrix} S_4^{\mathrm{s.s.}}(X_j,X_k) & -\int_{X_j}^{X_k}S_4^{\mathrm{s.s.}}(X_j,u)\,\mathrm{d}u \\ \frac{\partial}{\partial X_j}S_4^{\mathrm{s.s.}}(X_j,X_k) & S_4^{\mathrm{s.s.}}(X_k,X_j)\end{bmatrix}_{j,k=1,\ldots,n} \tag{3.46}$$

where as in the derivation of (3.33) a factor $\pi X_k^2/N$ has been removed from each odd numbered column and a factor of $\pi X_j^2/N$ removed from each even numbered row. Substituting (3.46) in (1.14) gives that for the circular ensemble with orthogonal symmetry the n-point distribution function in the neighbourhood of the spectrum singularity is given by

$$\rho_n^{\mathrm{GC}}(X_1,\ldots,X_n) = \mathrm{qdet}\begin{bmatrix} S_4^{\mathrm{s.s.}}(X_j,X_k) & -\int_{X_j}^{X_k}S_4^{\mathrm{s.s.}}(X_j,u)\,\mathrm{d}u \\ \frac{\partial}{\partial X_j}S_4^{\mathrm{s.s.}}(X_j,X_k) & S_4^{\mathrm{s.s.}}(X_k,X_j)\end{bmatrix}_{j,k=1,\ldots,n}. \tag{3.47}$$

4. Properties of ρ_n^{GC}

In the case $a = 0$ of the ensemble (1.8)—which corresponds to the Dyson circular ensemble—the scaled n-point distributions for $\beta = 1, 2$ and 4 were computed by Dyson [3] in his pioneering paper on quaternion determinants in random matrix theory. For $\beta = 1$ they are given by (3.34) with $S_1^{\mathrm{s.s.}}$ replaced by

$$S_1^{\mathrm{bulk}}(X,Y) := \frac{\sin\pi(X-Y)}{\pi(X-Y)} \tag{4.1}$$

while for $\beta = 4$ they are given by (3.47) with $S_4^{\mathrm{s.s.}}$ replaced by

$$S_4^{\mathrm{bulk}}(X,Y) := \frac{\sin 2\pi(X-Y)}{2\pi(X-Y)}. \tag{4.2}$$

Thus it must be that with $a = 0$, $S_1^{\mathrm{s.s.}}$ reduces to (4.1) while $S_4^{\mathrm{s.s.}}$ reduces to (4.2). According to (3.30), for $\beta = 1$,

$$\begin{aligned} S_1^{\mathrm{s.s}}(X,Y)\big|_{a=0} &= K^{\mathrm{s.s.}}(X,Y)\big|_{a=0} + \frac{\pi}{\Gamma(1/2)}\frac{J_{a+1/2}(\pi Y)}{(2\pi Y)^{1/2}}\left(1 - \lim_{a\to 0} a\int_0^{\pi X} s^{1-a}\,\mathrm{d}s\right) \\ &= K^{\mathrm{s.s.}}(X,Y)\big|_{a=0} \end{aligned} \tag{4.3}$$

where in obtaining the first equality use has been made of the facts that $1/\Gamma(a/2) \sim a/2$ as $a \to 0$, and $J_{a+1/2}(s) \sim (2s)^a/\Gamma(a+1/2)$ as $s \to 0$, while for $\beta = 4$ (3.44) gives

$$S_4^{\rm bulk}(X,Y)\big|_{a=0} = K(2X,2Y)\big|_{a=0}. \tag{4.4}$$

But it follows immediately from the definition (1.4) and the Bessel function formulae

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\sin x \qquad J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\cos x \tag{4.5}$$

that

$$K^{\rm s.s.}(X,Y)\big|_{a=0} = \frac{\sin\pi(X-Y)}{\pi(X-Y)} \tag{4.6}$$

so indeed

$$S_1^{\rm s.s.}(X,Y)\big|_{a=0} = S_1^{\rm bulk}(X,Y) \qquad S_4^{\rm s.s.}(X,Y)\big|_{a=0} = S_4^{\rm bulk}(X,Y). \tag{4.7}$$

Also related to the Dyson circular ensemble is the case $a = 1$ of the generalized circular ensemble. Thus the latter corresponds to fixing an eigenvalue at $\theta = 0$ in the former, so we must have

$$\rho_{n+1}^{\rm bulk}(X_1,\ldots,X_n,0) = \rho_n^{\rm s.s.}(X_1,\ldots,X_n)\big|_{a=1} \tag{4.8}$$

(this identity was first noted in [7] and checked explicitly for $\beta = 2$ using (1.3)). In the case $n = 1$, according to (3.34) and (3.47) the right-hand side of this expression is given by $S_\beta^{\rm s.s.}(X_1,X_1)\big|_{a=1}$ for $\beta = 1$ and 4 respectively. Recalling the known analytic form of $\rho_2^{\rm bulk}(X_1,0)$ [11] for these values of β we must therefore have

$$S_1^{\rm s.s.}(x,x)\big|_{a=1} = 1 - \frac{\sin^2\pi x}{(\pi x)^2} + \frac{1}{\pi}\left(\frac{\rm d}{{\rm d}x}\frac{\sin\pi x}{\pi x}\right)\left(-\frac{\pi}{2}{\rm sgn}\,x + \int_0^{\pi x}\frac{\sin t}{t}\,{\rm d}t\right) \tag{4.9}$$

$$S_4^{\rm s.s.}(x,x)\big|_{a=1} = 1 - \frac{\sin^2 2\pi x}{(2\pi x)^2} + \frac{1}{2\pi}\left(\frac{\rm d}{{\rm d}x}\frac{\sin 2\pi x}{2\pi x}\right)\int_0^{2\pi x}\frac{\sin t}{t}\,{\rm d}t. \tag{4.10}$$

Let us check the validity of (4.10) using (3.44) ((4.9) can be checked using similar working starting with (3.26)). Now, it follows from (3.27) that

$$\begin{aligned} K^{\rm s.s.}(2X,2X)\big|_{a=2} &= K^{\rm s.s.}(2X,2X)\big|_{a=0} - \frac{3(J_{3/2}(2\pi X))^2}{4\pi X} - \frac{(J_{1/2}(2\pi X))^2}{4\pi X} \\ &= 1 - \frac{\sin^2 2\pi X}{(2\pi X)^2} - \frac{3(J_{3/2}(2\pi X))^2}{4\pi X} \end{aligned} \tag{4.11}$$

where in obtaining the second equality use has been made of (4.5) and (4.6). Also, integrating by parts and using the Bessel function identity

$$\frac{\rm d}{{\rm d}s}J_\alpha(s) = J_{\alpha-1}(s) - \frac{\alpha}{s}J_\alpha(s) \tag{4.12}$$

with $\alpha = 3/2$ shows

$$\int_0^u s^{-1/2}J_{5/2}(s)\,{\rm d}s = -\frac{3J_{3/2}(u)}{2u} + 2\int_0^u \frac{J_{1/2}(s)}{s^{1/2}}\,{\rm d}s.$$

Substituting this result and (4.11) in (3.44) gives

$$S_4^{\rm s.s.}(x,x)\big|_{a=1} = 1 - \frac{\sin^2 2\pi x}{(2\pi x)^2} - 2\pi\frac{J_{3/2}(2\pi x)}{(2\pi x)^{1/2}}\int_0^{2\pi x}\frac{J_{1/2}(s)}{s^{1/2}}\,{\rm d}s$$

which is seen to be identical to (4.10) after substituting for $(2\pi x)^{-1/2}J_{3/2}(2\pi x)$ using (3.28), and then substituting for $J_{1/2}$ using (4.5).

Next we consider a scaled $a \to \infty$ limit of (3.26) and (3.44). Here, analogous to the situation with the hard edge distributions [6], we expect to connect to the corresponding soft edge distributions. The latter are specified by (3.34) and (3.47) but with $S_1^{\rm s.s.}$ and $S_4^{\rm s.s.}$ replaced by [6]

$$S_1^{\rm soft}(X, Y) = K^{\rm soft}(X, Y) + \tfrac{1}{2}{\rm Ai}(Y)\left(1 - \int_X^\infty {\rm Ai}(t)\,{\rm d}t\right) \tag{4.13}$$

$$S_4^{\rm soft}(X, Y) = K^{\rm soft}(X, Y) - \tfrac{1}{2}{\rm Ai}(Y)\int_X^\infty {\rm Ai}(t)\,{\rm d}t \tag{4.14}$$

where

$$K^{\rm soft}(X, Y) = \frac{{\rm Ai}(X){\rm Ai}'(Y) - {\rm Ai}(Y){\rm Ai}'(X)}{X - Y}$$

with ${\rm Ai}(x)$ denoting the Airy function. Now we know the Bessel function is related to the Airy function via the asymptotic expansion

$$J_a(x) \sim \left(\frac{2}{a}\right)^{1/3} {\rm Ai}\left(\frac{2^{1/3}(a - x)}{x^{1/3}}\right) \tag{4.15}$$

valid for a and x large and such that the argument of the Airy function is of order one. Noting from the Bessel function identity (4.12) that we can rewrite (1.4) as

$$\frac{1}{\pi}K^{\rm s.s.}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) = \frac{\phi(x)y\phi'(y) - \phi(y)x\phi'(x)}{x - y} \tag{4.16}$$

where

$$\phi(x) := \sqrt{\frac{x}{2}}J_{a+1/2}(x)$$

we can check using (4.15) that

$$\lim_{a\to\infty} -\left(\frac{a}{2}\right)^{1/3}\frac{1}{\pi}K^{\rm s.s.}\left(\frac{1}{\pi}(a - (a/2)^{1/3}x), \frac{1}{\pi}(a - (a/2)^{1/3}y)\right) = K^{\rm soft}(x, y). \tag{4.17}$$

Use of (4.17) and further use of (4.15) in (3.26) and (3.44) then shows

$$\lim_{a\to\infty} -\left(\frac{a}{2}\right)^{1/3}\frac{1}{\pi}S_1^{\rm s.s.}\left(\frac{1}{\pi}(a - (a/2)^{1/3}x), \frac{1}{\pi}(a - (a/2)^{1/3}y)\right) = S_1^{\rm soft}(x, y) \tag{4.18}$$

$$\lim_{a\to\infty} -\left(\frac{a}{2}\right)^{1/3}\frac{1}{2\pi}S_4^{\rm s.s.}\left(\frac{1}{2\pi}(a - (a/2)^{1/3}x), \frac{1}{2\pi}(a - (a/2)^{1/3}y)\right) = S_4^{\rm soft}(x, y) \tag{4.19}$$

which are the required connection formulae.

Our last property to be considered is motivated by the fact that the PDFs (1.2) and (1.8) have interpretations as being proportional to the Boltzmann factor of the one-component log-gas on a line and circle respectively, with an impurity charge of strength a at the origin. A log-gas is an example of a Coulomb system. In the thermodynamic limit the latter have the neutrality property that the total excess charge density about a fixed charge will be equal and opposite to that of the charge (see e.g. [10]). Now for a one-component log-gas with unit positive charges the charge density is the same as the particle density. The neutrality property at the fixed charge of strength a then implies the sum rule

$$2\int_0^\infty \big(\rho_1(x) - 1\big)\,{\rm d}x = -a \tag{4.20}$$

(the factor of 2 comes from the symmetry of the charge excess about the origin, while we subtract 1 from $\rho_1(x)$ since it is the unperturbed density). But $\rho_1(x) = S_\beta(x, x)$ ($\beta = 1, 2, 4$ with $S_2 := K^{\text{s.s.}}$) so we must have

$$2\int_0^\infty \big(S_\beta(x,x) - 1\big)\,\mathrm{d}x = -a. \tag{4.21}$$

We will use the exact expressions for $S_\beta(x, x)$ to verify (4.21) for each of the couplings $\beta = 1, 2, 4$, starting with $\beta = 2$. For $\beta = 2$, we see by taking the limit $y \to x$ in (1.4) that

$$S_2(x,x) = \frac{\pi^2}{2}x[J'_{a+(1/2)}(\pi x)J_{a-(1/2)}(\pi x) - J_{a+(1/2)}(\pi x)J'_{a-(1/2)}(\pi x)].$$

We make use of the Bessel function identities

$$\pi x J'_{a+(1/2)}(\pi x) = -\big(a + \tfrac{1}{2}\big)J_{a+(1/2)}(\pi x) + \pi x J_{a-(1/2)}(\pi x)$$
$$\pi x J'_{a-(1/2)}(\pi x) = \big(a - \tfrac{1}{2}\big)J_{a-(1/2)}(\pi x) - \pi x J_{a+(1/2)}(\pi x)$$

to rewrite this expression as

$$S_2(x,x) = -\pi a J_{a+(1/2)}(\pi x)J_{a-(1/2)}(\pi x) + \frac{\pi^2}{2}x\big\{\big(J_{a-(1/2)}(\pi x)\big)^2 + \big(J_{a+(1/2)}(\pi x)\big)^2\big\}. \tag{4.22}$$

Using the integration formula [4]

$$\int_0^\infty J_\nu(cx)J_{\nu-1}(cx)\,\mathrm{d}x = \frac{1}{2c}$$

we conclude from (4.22) that

$$\int_0^\infty \big(S_2(x,x) - 1\big)\,\mathrm{d}x = -\frac{a}{2} + \frac{1}{2}\int_0^\infty \left(x(J_{a-(1/2)}(x))^2 + x(J_{a+(1/2)}(x))^2 - \frac{2}{\pi}\right)\mathrm{d}x. \tag{4.23}$$

Next we use the fact that

$$\int_0^X x\big((J_\nu(x))^2 + (J_{\nu+1}(x))^2\big)\,\mathrm{d}x = \frac{X^2}{2}\big((J_\nu(X))^2 + (J_{\nu+1}(X))^2$$
$$-J_{\nu-1}(X)J_{\nu+1}(X) - J_\nu(X)J_{\nu+2}(X)\big)$$

and then employ the large-X asymptotic expansion of the Bessel function to conclude

$$\int_0^X \left(x(J_{a-(1/2)}(x))^2 + x(J_{a+(1/2)}(x))^2 - \frac{2}{\pi}\right)\mathrm{d}x \underset{X\to\infty}{\sim} \mathrm{O}\left(\frac{1}{X}\right).$$

The integral in (4.23) thus vanishes and so (4.21) is verified for $\beta = 2$.

With (4.21) established at $\beta = 2$, we see from (3.30) that the validity of (4.21) at $\beta = 1$ is equivalent to the integration formula

$$\int_0^\infty \mathrm{d}t\,\frac{J_{a+1/2}(t)}{t^{1/2}}\left(1 - 2^{3/2}\frac{\Gamma(a/2+3/2)}{\Gamma(a/2)}\int_0^t \mathrm{d}s\, s^{-3/2}J_{a+1/2}(s)\right) = 0. \tag{4.24}$$

To verify (4.24) we make use of the definite integral [8]

$$\int_0^\infty \frac{J_{a+1/2}(t)}{t^{1/2}}\,\mathrm{d}t = \frac{\Gamma(a/2+1/2)}{2^{1/2}\Gamma(a/2+1)} \tag{4.25}$$

to evaluate the first term, and change variables $s \mapsto ts$ in the second to rewrite it as

$$-2^{3/2}\frac{\Gamma(a/2+3/2)}{\Gamma(a/2)}\int_0^1 \mathrm{d}s\,\frac{1}{s^{3/2}}\left(\int_0^\infty \mathrm{d}t\,\frac{1}{t}J_{a+1/2}(t)J_{a+1/2}(ts)\right). \tag{4.26}$$

But for $0 < s < 1$ [4],

$$\int_0^\infty \frac{1}{t}J_{a+1/2}(t)J_{a+1/2}(ts)\,\mathrm{d}t = \frac{s^{a+1/2}}{2a+1}$$

so we see that (4.26) is equal to minus (4.25), thus verifying (4.24).

It remains to verify (4.21) in the case $\beta = 4$. Now we see from (3.44) that knowledge of the validity of (4.21) at $\beta = 2$ means it suffices to verify the integration formula

$$\int_0^\infty \mathrm{d}t\, \frac{J_{2a-1/2}(2t)}{t^{1/2}} \int_0^t \mathrm{d}s\, \frac{J_{2a+1/2}(2s)}{s^{1/2}} = 0. \tag{4.27}$$

Changing variables $s \mapsto ts$ and interchanging the integration order shows that the integral can be written

$$\frac{1}{2}\int_0^1 \frac{\mathrm{d}s}{s^{1/2}}\left(\int_0^\infty \mathrm{d}t\, J_{2a-1/2}(t)J_{2a+1/2}(st)\right). \tag{4.28}$$

But for $0 < s < 1$ the integral over t in (4.28) vanishes [4] and consequently (4.27) is verified.

Acknowledgment

P J Forrester was supported by the ARC.

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